The Weyl–Kac weight formula

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Abstract. We provide the first formulae for the weights of all simple highest weight modules over Kac–Moody algebras. For generic highest weights, we present a formula for the weights of simple modules similar to the Weyl–Kac character formula. For the remaining highest weights, the formula fails in a striking way, suggesting the existence of 'multiplicity-free' Macdonald identities for affine root systems.

Keywords: Macdonald identity, Weyl–Kac formula, Kac–Moody algebra, highest weight module

1 Introduction

The finite dimensional simple representations of complex semisimple Lie algebras have long been objects of central concern in algebraic combinatorics. For a semisimple Lie algebra $\mathfrak{g}$, with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and Weyl group W, such representations are parametrized by weights λ in the dominant Weyl chamber in $\mathfrak{h}^*$. The character of the corresponding simple module $L(\lambda)$ is given by the Weyl character formula [27, 28, 29]. If for $\mu \in \mathfrak{h}^*$, we write $L(\lambda)_\mu$ for the corresponding weight space, and writing Δ^+ for the weights of $\mathfrak{n}^+$, i.e. the positive roots, and ρ for half their sum, the formula reads as:

$$\sum_{\mu \in \mathfrak{h}^*} \dim L(\lambda)_\mu e^\mu = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}. \tag{1.1}$$

We wish to remark on two points at this juncture. First, there exist positive combinatorial formulae for $\dim L(\lambda)_\mu$, coming from the theory of crystal bases. For example, for $\mathfrak{g}$ of type A, these characters are essentially Schur polynomials, and weight space multiplicities are given by counting semistandard Young tableaux. Second, for the more basic question of which $\dim L(\lambda)_\mu$ are positive, there are two very simple descriptions. Indeed, one can (i) take the convex hull of the Weyl group orbit $W\lambda$, and intersect this

with an appropriate translate of the root lattice to obtain the nonzero weights. Alternatively, (ii) by W invariance, it suffices to consider dominant μ, for which $L(\lambda)_\mu$ is positive if and only if $\lambda - \mu$ is a sum of positive roots.

When we consider a general simple highest weight module $L(\lambda)$, i.e. we no longer assume λ is dominant integral, the character is known through Kazhdan–Lusztig theory [16, 6, 4]. For example, if $\lambda = y(\nu + \rho) - \rho$, for ν dominant integral, and $y \in W$, we have:

$$\sum_{\mu \in \mathfrak{h}^*} \dim L(\lambda)_\mu e^\mu = \frac{\sum_{w \in W} (-1)^{\ell(w)} P_{wyw_\circ, yw_\circ}(1) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}. \tag{1.2}$$

Here $P_{x,y}(1)$ denotes the Kazhdan–Lusztig (KL) polynomial evaluated at 1, and $w_\circ$ the longest element of W. What is relevant for us is that as in the case of $y = 1$, i.e. the Weyl character formula, there is cancellation on the right hand side, and the situation is made more subtle by the appearance of KL polynomials.

Unlike for finite dimensional simple modules, positive combinatorial formulae for the multiplicities of weight spaces $\dim L(\lambda)_\mu$ are not known in general. This is a problem of longstanding interest, and there are partial results, for example due to Mathieu and Papadopoulo in type A [21]. Surprisingly, even the simpler question of describing which multiplicities are positive was not known until recent work of the second author [17].

The infinite dimensional cousins of semisimple Lie algebras, the affine Lie algebras, also play a celebrated role in algebraic combinatorics. In broad strokes, the combinatorial representation theory of affine Lie algebras proceeds similarly to that of semisimple Lie algebras.

Integrable highest weight modules are the analogues of finite dimensional simple modules, and their formal characters are given by the Weyl–Kac formula, a modification of (1.1). For example, it was famously realized by Kac that Macdonald's identities for affine root systems are precisely the Weyl–Kac formula specialized to the trivial modules of untwisted affine algebras [11]. As in the first remark following (1.1), weight multiplicities in integrable highest weight modules can be obtained positively from combinatorial descriptions of crystal bases. A remarkable example is the Young graph of p-regular partitions, which is the crystal graph of the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$, for any prime p [10, 22]. As in the second remark following (1.1), it is again true that the locus of weights with positive multiplicity admits two simple descriptions similar to (i), (ii) above.

For nonintegrable simple highest weight modules $L(\lambda)$, as in finite type the formal characters are again largely understood. However, the answer is considerably subtler. For integral λ of positive or negative level, the characters are determined by formulae similar to (1.2), now involving inverse Kazhdan–Lusztig polynomials and Kazhdan–Lusztig polynomials for the affine Weyl group, respectively [13, 14]. For λ at the *critical* level, which plays a distinguished role in the Geometric Langlands program [9], the Feigin–Frenkel conjecture predicts a formula involving *periodic* Kazhdan–Lusztig poly-

nomials, and a modified Weyl denominator [1]. As for semisimple Lie algebras, so far individual weight multiplicities have resisted combinatorial interpretation; moreover, even the simpler question of characterizing which multiplicities are positive was unanswered.

In this extended abstract of the paper [7], we explain a solution to this latter problem. We will present three positive formulae for the weights of arbitrary simple modules over semisimple and affine Lie algebras. We will then provide a formula for the weights of all nontrivial simple modules that is strikingly similar to the Weyl–Kac formula, i.e. involving signs but no Kazhdan–Lusztig polynomials or their variants. The truth of the formula for nontrivial modules suggests extensions of the formulae of Brion and Khovanskii–Pukhlikov for exponential sums over lattice points of (virtual) polyhedra. The failure of the formula for trivial modules suggests the existence of 'multiplicity-free' Macdonald identities.

2   Three formulae for the weights of simple modules

Before stating our results, let us establish some context for non-experts. Semisimple and affine Lie algebras both admit transparent presentations by generators and relations which can be read off of the Dynkin diagram of the corresponding root system. Such Lie algebras built from Dynkin diagrams are known as Kac–Moody algebras. If I is the set of nodes of the Dynkin diagram, then for each $i \in I$ one has operators f_i, h_i, e_i. What is relevant here is that the h_i pairwise commute, and lie in a maximal commutative Lie subalgebra $\mathfrak{h}$. The f_i, e_i, often called lowering and raising operators, respectively, are simultaneous eigenvectors for the adjoint action of $\mathfrak{h}$, with opposite eigenvalues $-\alpha_i, \alpha_i \in \mathfrak{h}^*$, respectively. The eigenvalues $\alpha_i, i \in I$, are called the positive simple roots.

Let $\mathfrak{g}$ be a Kac–Moody algebra. A $\mathfrak{g}$-module V is called *highest weight* if it can be generated by a single vector $v \in V$ that behaves very simply under the action of 'two thirds' of the generators: (i) $hv = (h, \lambda)v$, for some $\lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$, and (ii) $e_i v = 0, \forall i \in I$.[3] Such a vector v is then automatically unique up to scalars.

As in the introduction, the simple highest weight modules over $\mathfrak{g}$ are parametrized by their highest weight $\lambda \in \mathfrak{h}^*$, and we write $L(\lambda)$ for the corresponding simple module. We now present three formulae for the weights of $L(\lambda)$, which are defined to be:

$$\operatorname{wt} L(\lambda) := \{\mu \in \mathfrak{h}^* : \dim L(\lambda)_\mu > 0\}. \tag{2.1}$$

The first formula uses restriction to a Levi subalgebra corresponding a subdiagram of the Dynkin diagram. Specifically, write $I_\lambda := \{i \in I : (h_i, \lambda) \in \mathbb{Z}^{\geqslant 0}\}$. Write $\mathfrak{l}$ for the

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[3]To make condition (ii) seem more natural, it may be helpful to note that for a finite dimensional representation of a semisimple Lie algebra, the $e_i, i \in I$ always act nilpotently. Moreover, the dimension of their simultaneous kernel is the number of simple summands of the representation. Replacing condition (ii) by a 'character' in $(\mathfrak{n}^+/[\mathfrak{n}^+, \mathfrak{n}^+])^\vee$ leads to the *Whittaker* modules [19].

subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}, e_i, f_i, i \in I_\lambda$. For $\nu \in \mathfrak{h}^*$, write $L_\mathfrak{l}(\nu)$ for the simple highest weight module for $\mathfrak{l}$ of highest weight ν. Finally, for a subset $S \subset \mathfrak{h}^*$, write $\mathbb{Z}^{\geqslant 0} S \subset \mathfrak{h}^*$ for the set of nonnegative integral combinations of elements of S. The formula then reads as:

Theorem 2.2 (Integrable Slice Decomposition, Dhillon–Khare [7])**.**

$$\operatorname{wt} L(\lambda) = \bigsqcup_{\mu \in \mathbb{Z}^{\geqslant 0}\{\alpha_i, i \in I \setminus I_\lambda\}} \operatorname{wt} L_\mathfrak{l}(\lambda - \mu). \tag{2.3}$$

The usefulness of Theorem 2.2 is that $\mathfrak{l}$ is the direct sum of an abelian Lie algebra which acts semisimply on $L(\lambda)$ and the Kac–Moody algebra associated to the Dynkin diagram with nodes I_λ. Moreover, for the latter algebra each $L_\mathfrak{l}(\lambda - \mu)$ is an integrable highest weight module, for which the weights are well known.

We include two illustrations of Theorem 2.2 in Figure 2.1. In the lefthand example, the Levi $\mathfrak{l}$ contains a copy of $\mathfrak{sl}_3$ generated by e_1, e_2, f_1, f_2. As $3 \notin I_\lambda$, the weight spaces of $L(\lambda)$ corresponding to $\lambda, \lambda - \alpha_3, \lambda - 2\alpha_3$, etc. will all be nonzero. Moreover, they are highest weight with respect to $\mathfrak{sl}_3$, and generate finite dimensional $\mathfrak{l}$ representations. The convex hulls of weights of the first three of these representations appear here shaded. These are the 'integrable slices' of Theorem 2.2, and the theorem says these $\mathfrak{sl}_3$ representations produce all weights of $L(\lambda)$. The righthand example, and (2.3) in general, can be read similarly.

The second formula shows the relationship between $\operatorname{wt} L(\lambda)$ and its convex hull. We recall the standard partial order on weights, where $\nu \leqslant \nu'$ if and only if $\nu' - \nu$ is a nonnegative integral combination of positive simple roots. For a subset $X \subset \mathfrak{h}^*$, we write $\operatorname{conv} X$ for its convex hull. When $X = \operatorname{wt} L(\lambda)$, we abbreviate this to $\operatorname{conv} L(\lambda)$.

Theorem 2.4 (Dhillon–Khare [7])**.**

$$\operatorname{wt} L(\lambda) = \operatorname{conv} L(\lambda) \cap \{\mu \in \mathfrak{h}^* : \mu \leqslant \lambda\}. \tag{2.5}$$

The question of whether the weights of simple modules $L(\lambda)$ are no finer an invariant than their convex hull was raised by Daniel Bump. Theorem 2.4 answers this question affirmatively. Note the similarity of Theorem 2.4 to description (i) of the weights of simple finite dimensional modules mentioned in the introduction.

Theorem 2.4 is complemented by the following description of the convex hull of a simple highest weight module. We recall that the Weyl group W of $\mathfrak{g}$ is generated by reflections in $\mathfrak{h}^*$ indexed by nodes of the Dynkin diagram $s_i, i \in I$. Let us write W_{I_λ} for the subgroup generated by $s_i, i \in I_\lambda$.

Proposition 2.6 (Dhillon–Khare [7])**.**

$$\operatorname{conv} L(\lambda) = \operatorname{conv} \bigcup_{w \in W_{I_\lambda}, i \in I \setminus I_\lambda} w\{\lambda - \mathbb{Z}^{\geqslant 0}\alpha_i\}. \tag{2.7}$$

When $I_\lambda = I$, by the right hand side we mean $\bigcup_{w \in W} w\lambda$.
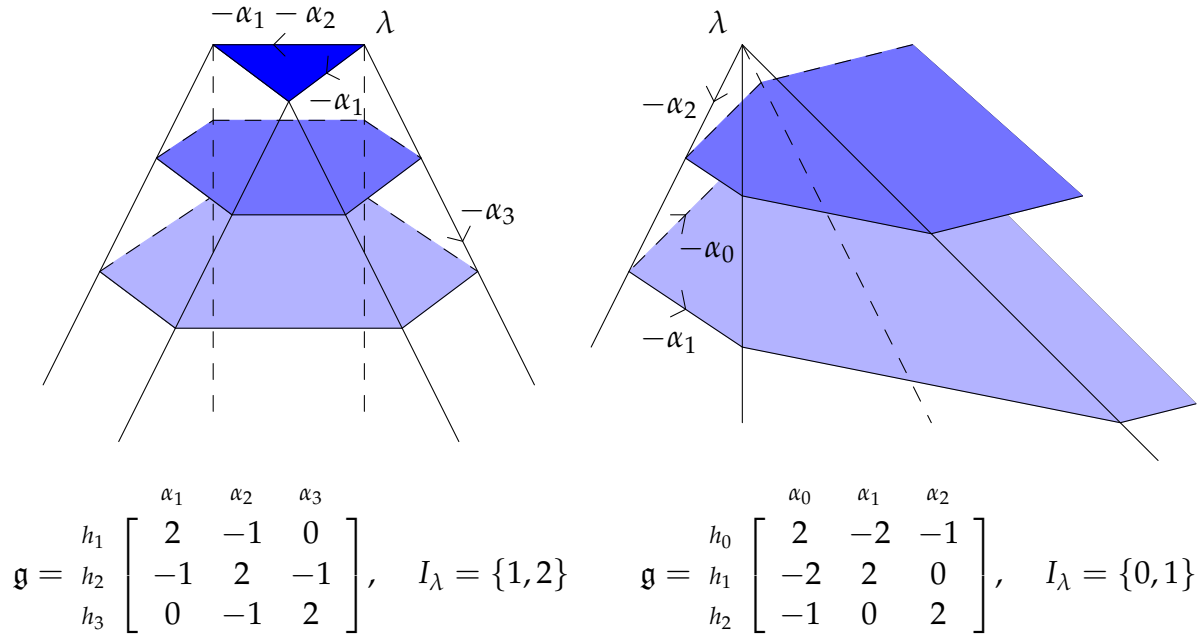
$$\mathfrak{g} = \begin{array}{c} \\ h_1 \\ h_2 \\ h_3 \end{array} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \left[\begin{array}{ccc} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] \end{array}, \quad I_\lambda = \{1,2\} \qquad \mathfrak{g} = \begin{array}{c} \\ h_0 \\ h_1 \\ h_2 \end{array} \begin{array}{ccc} \alpha_0 & \alpha_1 & \alpha_2 \\ \left[\begin{array}{ccc} 2 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{array}\right] \end{array}, \quad I_\lambda = \{0,1\}$$

Figure 2.1: Two illustrations of Theorem 2.2, with finite and infinite integrability. Note we only draw three of the infinitely many 'integrable slices', or rather the convex hulls of their weights, in each figure (in the righthand figure, the top slice is a point). Similarly, we only include four of the infinitely many rays leaving λ in the righthand figure.

The third formula uses the Weyl group action on $\mathfrak{h}^*$. We will use the following parabolic analogue of the dominant chamber:

$$P_\lambda^+ := \{\nu \in \mathfrak{h}^* : (h_i, \nu) \in \mathbb{Z}^{\geqslant 0}, \forall i \in I_\lambda\}.$$

Then the final formula reads as:

Theorem 2.8 (Dhillon–Khare [7])**.** *Suppose λ has finite stabilizer in W_{I_λ}. Then:*

$$\operatorname{wt} L(\lambda) = W_{I_\lambda}\{\mu \in P_\lambda^+ : \mu \leqslant \lambda\}. \tag{2.9}$$

Note the similarity of Theorem 2.8 to description (ii) of the weights of finite dimensional modules appearing in the introduction. Let us comment on the hypothesis on λ appearing in Theorem 2.8, as it will reappear shortly. The assumption is met by all $L(\lambda)$ for semisimple Lie algebras. For $\mathfrak{g}$ an affine algebra with connected Dynkin diagram, let us call a simple module $L(\lambda)$ *trivial* if $\dim L(\lambda) = 1$, or equivalently $(h_i, \lambda) = 0, \forall i \in I$. Then the assumption is met by all $L(\lambda)$ which are not trivial.

As stated, Theorem 2.8 does not hold for all λ. For experts, this can be seen by thinking about a trivial module for $\widehat{\mathfrak{sl}}_2$ and lowering by an imaginary root. However,

as we explain in [8], it can be corrected by using a refinement of the partial order $\leqslant$ introduced by Kac and Peterson [12].

For integrable $L(\lambda)$, Theorem 2.2 is a tautology, and Theorems 2.4 and 2.8 are well known. Theorems 2.2 and 2.4 are due to the second named author for semisimple Lie algebras [17]. All other cases are to our knowledge new.

The above descriptions of $\operatorname{wt} L(\lambda)$ are particularly striking in infinite type. For $\mathfrak{g}$ affine, the formulae are uniform across the negative, critical, and positive levels, in contrast to the (partly conjectural) character formulae discussed in the introduction. When $\mathfrak{g}$ is symmetrizable[4], we similarly obtain weight formulae at critical λ. The authors are unaware of even conjectural formulae for $\operatorname{ch} L(\lambda)$ in this case. Finally, when $\mathfrak{g}$ is non-symmetrizable it is a notoriously difficult problem to compute exact multiplicities. To wit, it is completely unknown how to compute weight space multiplicities for integrable $L(\lambda)$ or even the adjoint representation. Thus for $\mathfrak{g}$ non-symmetrizable, Theorems 2.2, 2.4 and 2.8 provide as much information on $\operatorname{ch} L(\lambda)$ as one could hope for, given existing methods.

3   The Weyl–Kac weight formula

We now turn to a formula for $\operatorname{wt} L(\lambda)$ similar to the Weyl–Kac character formula (recall that $\operatorname{wt} L(\lambda)$ was defined in (2.1)). To orient ourselves, we first remind a slightly nonstandard presentation of the Weyl–Kac formula in Proposition 3.2 below. This presentation is due to Atiyah and Bott for semisimple Lie algebras [2]. As we indicate in [7], it can be straightforwardly adapted to Kac–Moody algebras.

We will need a little notation and a convention. The subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$ generated by the $e_i, i \in I$ is a semisimple $\mathfrak{h}$-module. Write $\Delta^+ \subset \mathfrak{h}^*$ for the weights of $\mathfrak{n}^+$. For $\alpha \in \Delta^+$, write $\mathfrak{g}_\alpha$ for the corresponding eigenspace. Next, for $w \in W$ and $\alpha \in \Delta^+$, by $w(1 - e^{-\alpha})^{-1}$ we mean the 'highest weight' expansion, i.e.:

$$w\frac{1}{1 - e^{-\alpha}} := \begin{cases} 1 + e^{-w\alpha} + e^{-2w\alpha} + \cdots, & w\alpha > 0, \\ -e^{w\alpha} - e^{2w\alpha} - e^{3w\alpha} - \cdots, & w\alpha < 0. \end{cases} \tag{3.1}$$

With these preliminaries, we may state:

Proposition 3.2. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, and $L(\lambda)$ an integrable highest weight module, i.e. $I_\lambda = I$. Then:*

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} w\frac{e^\lambda}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \tag{3.3}$$

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[4]For non-experts, this is a technical condition on the multiplicities of edges between nodes of the Dynkin diagram, which is automatic for semisimple and affine Lie algebras.

Of course, for nonintegrable $L(\lambda)$, the character is much more involved, as reminded in the introduction. It therefore may be surprising that the weights of generic $L(\lambda)$ admit a similar description. To do so, let us package the weights into a multiplicity-free character:

$$\operatorname{wt} L(\lambda) = \sum_{\substack{\nu \in \mathfrak{h}^*: \\ L(\lambda)_\nu \neq 0}} e^\nu.$$

We then have:

Theorem 3.4 (Dhillon–Khare [7]). *Let $\mathfrak{g}$ be an arbitrary Kac–Moody algebra. If the stabilizer of λ in W_{I_λ} is finite, then:*

$$\operatorname{wt} L(\lambda) = \sum_{w \in W_{I_\lambda}} w \frac{e^\lambda}{\prod_{i \in I}(1 - e^{-\alpha_i})}. \tag{3.5}$$

Note the similarity of (3.3) and (3.5). Theorem 3.4 was known in the case of $L(\lambda)$ integrable [5, 15, 23, 24, 26]. All other cases are to our knowledge new, even in finite type.

For $\mathfrak{g}$ a semisimple Lie algebra, the application of Theorem 3.4 to $L(0)$ is seen to be equivalent to the following identity, which may be thought of as a 'coordinate-free' denominator identity:

Corollary 3.6 (Dhillon–Khare [7]). *Let Δ be a finite root system. Let Π denote the set of all bases for Δ, cf. [25]. Then:*

$$\prod_{\alpha \in \Delta}(1 - e^{-\alpha}) = \sum_{\pi \in \Pi} \prod_{\beta \in \Delta \setminus \pi}(1 - e^{-\beta}). \tag{3.7}$$

3.1 Brion's formula beyond polyhedra

We wish to call the attention of the reader to two further problems suggested by Theorem 3.4.

Firstly, when the convex hull of $\operatorname{wt} L(\lambda)$ is a polyhedron, Theorem 3.4 may be obtained from Brion's formula. This is a more general formula for exponential sums over lattice points of polyhedra, due to Brion [5] for rational polytopes and Lawrence [20] and Khovanskii–Pukhlikov [18] in general, cf. [3]. We thank Michel Brion for bringing this to our attention. For experts, one needs to observe that for regular λ the associated polyhedron is Delzant. The case of singular λ may then be obtained via a deformation argument due to Postnikov [23].

For infinite dimensional $\mathfrak{g}$, the convex hulls $\operatorname{conv} L(\lambda)$ are rarely polyhedral, e.g. since the Weyl group is infinite and $w\lambda$ are all extremal points. Instead, one has the following result:

Theorem 3.8 (Dhillon–Khare [8])**.** *Let $\mathfrak{g}$ correspond to a connected Dynkin diagram of infinite type, and let $L(\lambda)$ be a nontrivial module, i.e. $\dim L(\lambda) > 1$. Then the following are equivalent:*

1. *The stabilizer of λ in W_{I_λ} is finite.*

2. *The convex hull of $\operatorname{wt} L(\lambda)$ is locally polyhedral, i.e. its intersection with every polytope is a polytope.*

The combination of Theorems 3.4 and 3.8 suggest that Brion's formula should be true for an appropriate class of locally polyhedral convex sets containing both polyhedra and sets of the form $\operatorname{conv} L(\lambda)$. This in particular would simplify the argument for Theorem 3.4 for integrable modules.

3.2 'Multiplicity-free' Macdonald identities

The second problem we wish to mention to the reader concerns the extension of Theorem 3.4 to trivial modules. For root systems of infinite type, the trivial module $L(0)$ no longer satisfies the condition of Theorem 3.4. Moreover, the stated equality no longer always holds, and instead fails in very striking ways. For example, we obtained the following identity by a direct calculation:

Proposition 3.9 (Dhillon–Khare [7])**.** *Let $\mathfrak{g}$ correspond to a Dynkin diagram of infinite type with two nodes. Write Δ_{im}^+ for the positive imaginary roots of $\mathfrak{g}$. Then:*

$$\sum_{w \in W} w \frac{1}{\prod_{i \in I}(1 - e^{-\alpha_i})} = 1 + \sum_{\delta \in \Delta_{im}^+} e^{-\delta}. \tag{3.10}$$

As in the Macdonald identities, i.e. the denominator identity for affine Lie algebras, the naive equality one would guess using only real roots turns out to require correction terms coming from the imaginary roots. Moreover, as we are deforming a 'multiplicity-free' denominator identity for finite root systems, i.e. for $\operatorname{wt} L(\lambda)$ rather than $\operatorname{ch} L(\lambda)$, the correction terms appearing in (3.10) are insensitive to the multiplicity of the imaginary root spaces $\mathfrak{g}_\delta$. It would be very interesting to obtain identities similar to (3.10) for higher rank infinite root systems, i.e. more 'multiplicity-free' Macdonald identities.

4   Some ingredients of the proofs

Having explained the statements of some of the results of [7], we now take a moment to highlight some new ingredients that went into their proofs. While the above statements concern the simple highest weight modules, they emerge from a study of general highest weight $\mathfrak{g}$-modules V.

For a fixed $\lambda \in \mathfrak{h}^*$, one might try to classify all the modules with highest weight λ. However, this turns out to be a daunting task: even for semisimple Lie algebras of low rank, such as $\mathfrak{sl}_5$, there can be infinitely many non-isomorphic highest weight modules of highest weight λ. One therefore can try to use invariants to distinguish between members of this profusion. The following theorem says that several of these invariants, seemingly different, are in fact the same:

Theorem 4.1 (Dhillon–Khare [7]). *Let $\mathfrak{g}$ be an arbitrary Kac–Moody algebra. Let V be a highest weight $\mathfrak{g}$-module. The following data are equivalent:*

1. *I_V, the integrability of V, i.e. $I_V = \{i \in I : f_i \text{ acts locally nilpotently on } V\}$.*

2. *conv V, the convex hull of the weights of V.*

3. *The stabilizer of ch V in W.*

To our knowledge, Theorem 4.1 is new even for semisimple Lie algebras. Before explaining how it connects to the problem of determining the weights of simple modules, let us mention a convexity-theoretic consequence.

Corollary 4.2 (Ray Decomposition, Dhillon–Khare [7]). *Let V be a highest weight module, and let I_V be as in Theorem 4.1. Let W_{I_V} denote the subgroup of W generated by $s_i, i \in I_V$. Then:*

$$\text{conv } V = \text{conv} \bigcup_{w \in W_{I_V}, i \in I \setminus I_V} w(\lambda - \mathbb{Z}^{\geqslant 0} \alpha_i) \tag{4.3}$$

When $I_V = I$, by the right hand side we mean $\bigcup_{w \in W} w\lambda$.

We include two illustrations of Corollary 4.2 in Figure 4.1. To our knowledge Corollary 4.2 was not previously known for non-integrable modules in either finite or infinite type.

Notice that Proposition 2.6 is a special case of Corollary 4.2. The above presentation of the convex hull can be understood as follows. Consideration of the nodes of $I \setminus I_V$, and the representation theory of $\mathfrak{sl}_2$ tell us that $\lambda - \mathbb{Z}^{\geqslant 0} \alpha_i$ lies in ch V, $\forall i \in I \setminus I_V$. Consideration of the nodes of I_V tell us the character, whence the convex hull, should be W_{I_V} invariant. The content of Corollary 4.2 is that these two *a priori* estimates are in fact enough to generate the convex hull.

As Corollary 4.2 indicates, Theorem 4.1 has useful implications in the convexity theoretic study of highest weight modules. In fact, in the companion work [8], we determine the face posets of all convex hulls of highest weight modules. To our knowledge, this classification had not been fully achieved even for semisimple Lie algebras.

Finally, let us explain how to use Theorem 4.1 to obtain information about wt $L(\lambda)$, i.e. how to pass from convex hulls to weights. For a highest weight module V, define
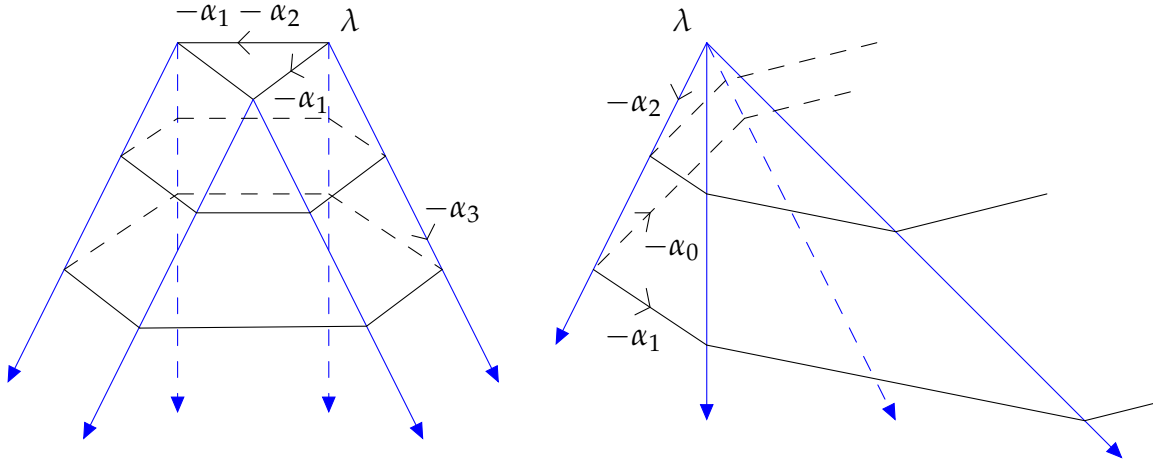
Figure 4.1: Ray Decomposition, with finite and infinite integrability; here $\mathfrak{g}$ and I_V are as in Figure 2.1.

the *potential integrability* of V to be $I_V^p := I_\lambda \setminus I_V$. To justify the terminology, note these are precisely the simple directions whose actions become integrable in quotients of V.

In the following theorem, we use the *parabolic Verma modules* $M(\lambda, J)$, for $J \subset I_\lambda$. These are characterized by the property that they have integrability J and admit a morphism to every highest weight module V of highest weight λ with $I_V = J$.

Theorem 4.4 (Dhillon–Khare [7]). *Let V be a highest weight module, V_λ its highest weight line. Let $\mathfrak{l}$ denote the Levi subalgebra generated by $\mathfrak{h}, e_i, f_i, i \in I_V^p$. Then $\operatorname{wt} V = \operatorname{wt} M(\lambda, I_V)$ if and only if $\operatorname{wt} U(\mathfrak{l})V_\lambda = \operatorname{wt} U(\mathfrak{l})M(\lambda, I_V)_\lambda$, i.e. $\operatorname{wt} U(\mathfrak{l})V_\lambda = \lambda - \mathbb{Z}^{\geqslant 0}\{\alpha_i, i \in I_V^p\}$.*

In particular, if $V = L(\lambda)$, or more generally $|I_V^p| \leqslant 1$, then $\operatorname{wt} V = \operatorname{wt} M(\lambda, I_V)$.

Theorem 4.4 is new in both finite and infinite type. Theorems 4.1 and 4.4, combined with some analysis of the parabolic Verma modules, yield proofs of the results advertised in the second and third sections. In fact, note that Theorems 2.2, 2.4 and 2.8 correspond to the manifestations 1., 2., and 3., respectively, of the invariant studied in Theorem 4.1.

Acknowledgements

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